Fluid dynamics in Computer Animation

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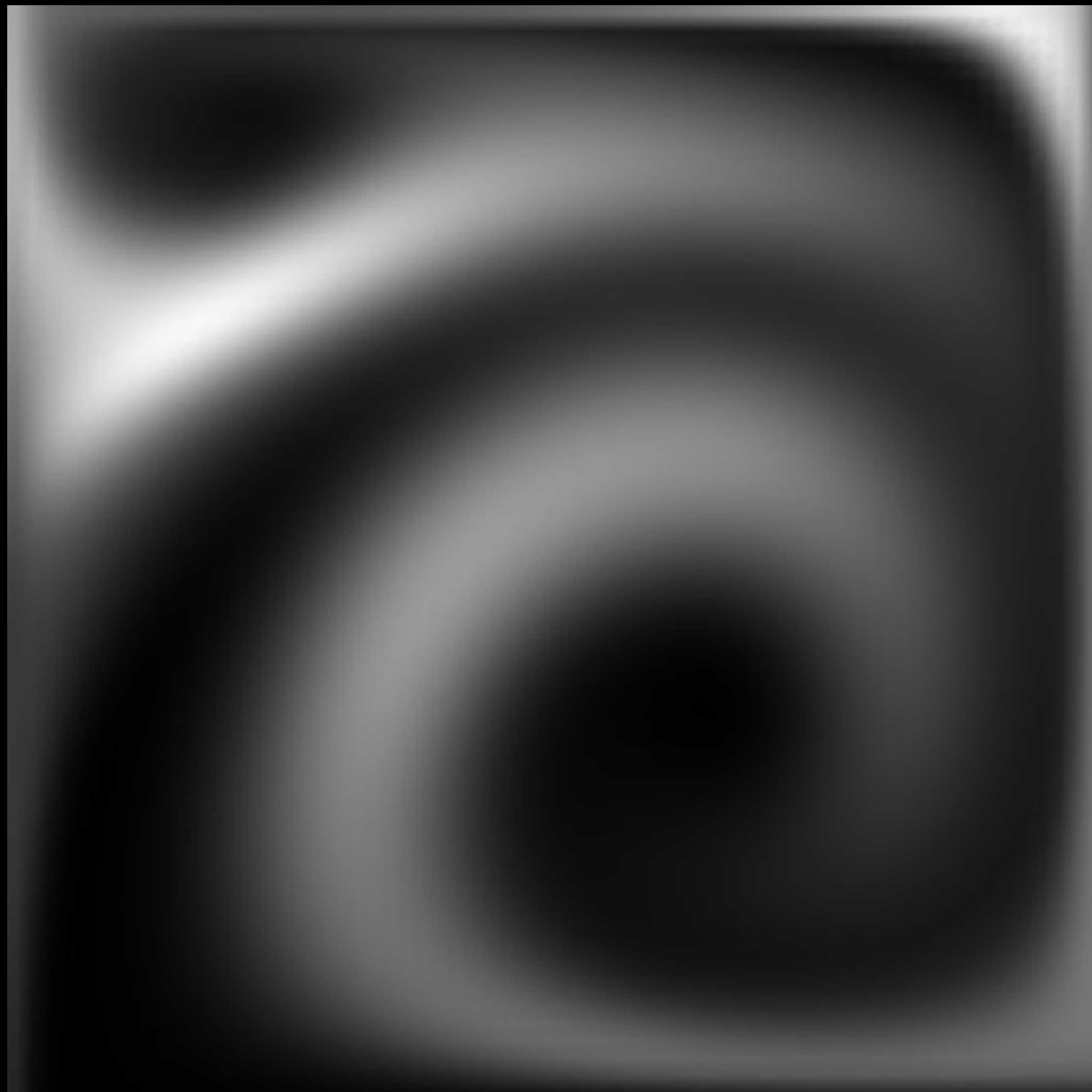
Talk overview

- FD: field of application
- physical & math background
- methods
- examples

Applications of FD

- Smoke
- Water
- Fire
- Viscous Fluids



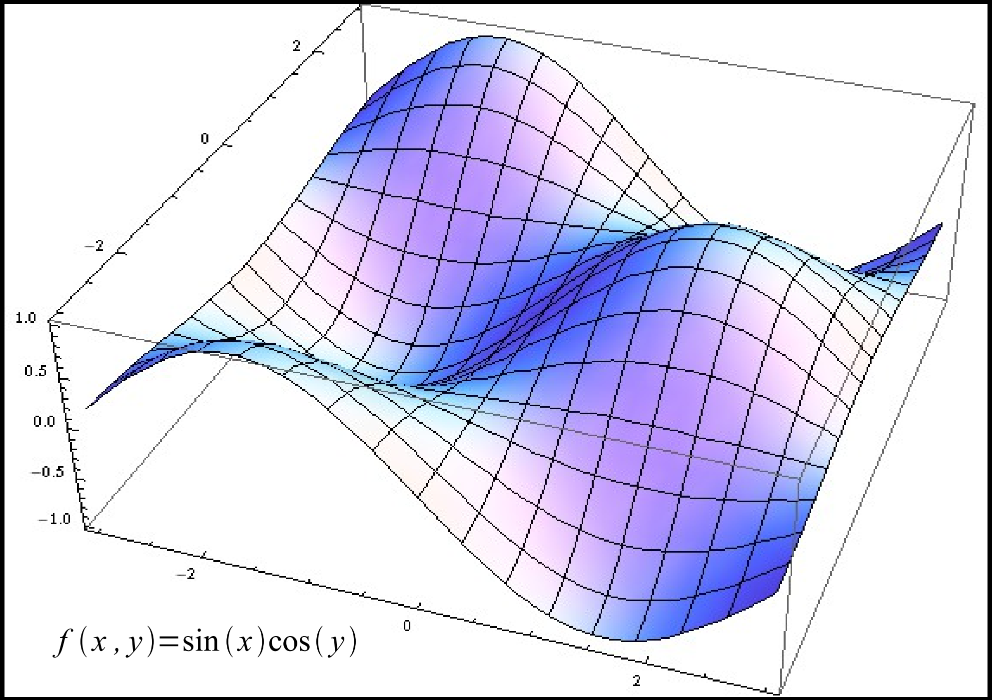
Math Phys background

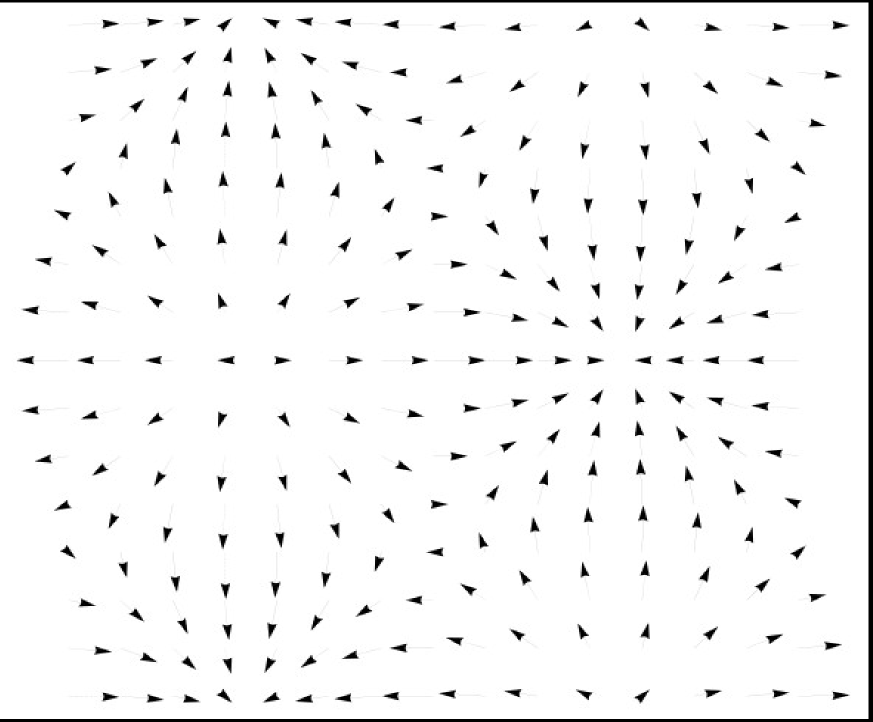
scalar field $f(x, y, z) \rightarrow \Re$

gradient $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = grad\ f$$

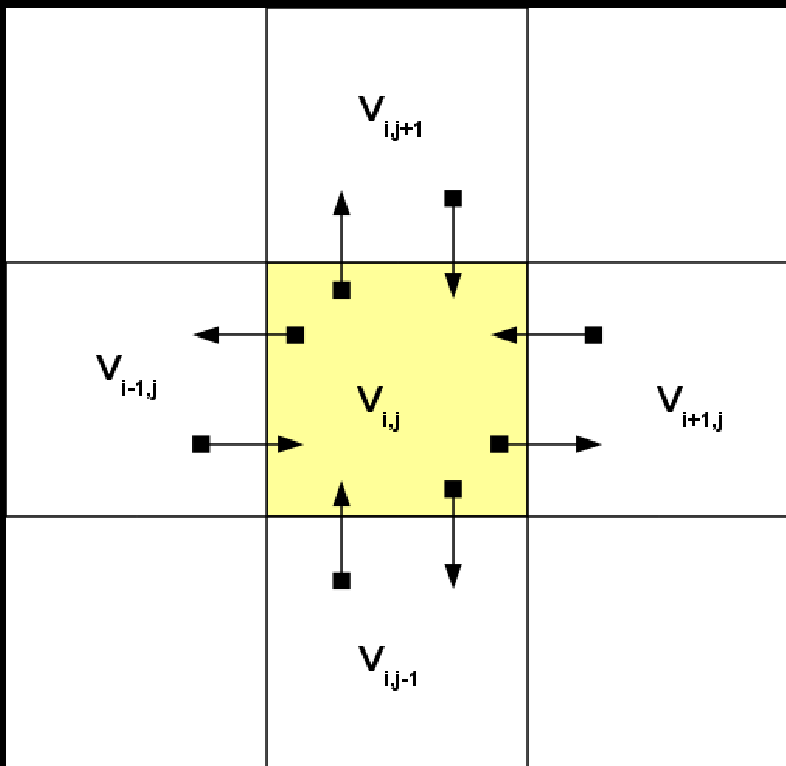
Math & Phys Background





$\boldsymbol{F} = \nabla f$

Math & Phys Background



Divergence

$$\nabla . \boldsymbol{F} = \mathrm{Div}\ \boldsymbol{F}$$

$$\nabla . \boldsymbol{u} = \nabla . (u, v, w) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

a measure of a flow through any infinitesimal volume dV surrounding a given point

Math & Phys Background

Curl

$$\nabla \times \boldsymbol{u} = \nabla \times (u, v, w) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

a measure of how vector field rotates around any point

Laplacian

$$\nabla . \nabla f = \nabla^2 \boldsymbol{f} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Physical Description of Fluids

Physics of FD: Euler, Navier, Stokes 1750-1850

Navier-Stokes Equations:

$$\nabla . \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} . \nabla)\boldsymbol{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

1950 > use of computers > numerical solutions

equations express the law of mass and momentum conservation

Density & Vector Fields

A Stable solver by Jos Stam

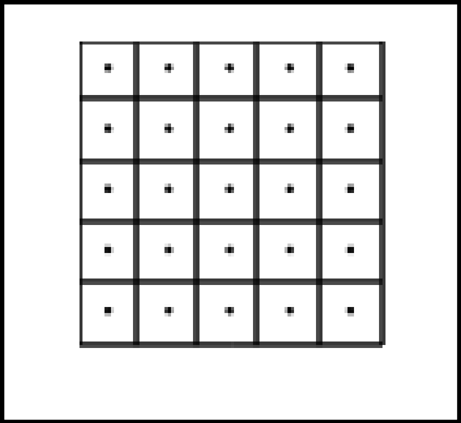
$$\frac{\partial \rho}{\partial t} = -(\boldsymbol{u} \, . \nabla)\rho + \kappa \nabla^2 \rho + S$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} \, . \nabla)\boldsymbol{u} + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

...solution is splitted into two parts given by contributions of density field and velocity field

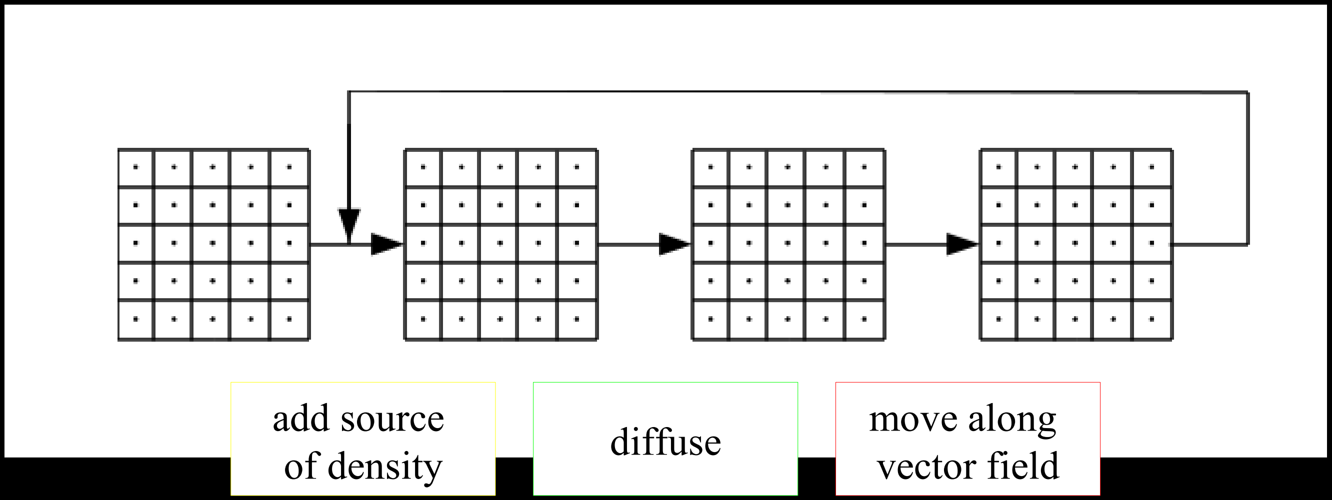
Solution Overview

```
Do {
        Get Force
        Get Density Source
        Update Velocity
        Update Density
        Display Density
} until simulation is in progress
```



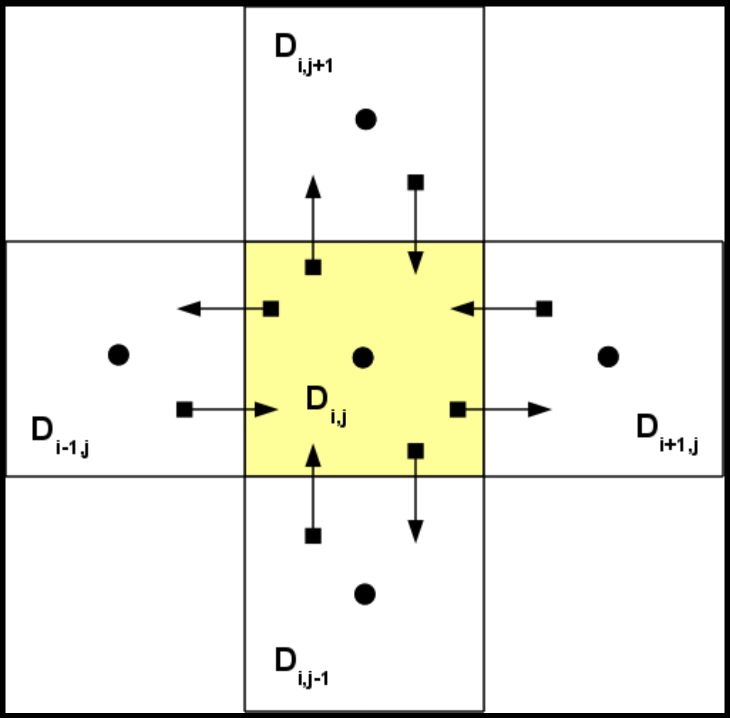
space subdivision

Density solver



$$\frac{\partial \rho}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\rho + \kappa \nabla^2 \rho + S$$

Diffusing the Density



diffusion on single border is given as difference of densities:

$$Diff = \kappa\, dt\, (P_{neighbor} - P_{center}) \frac{1}{h^2},\ h = \frac{1}{N}$$

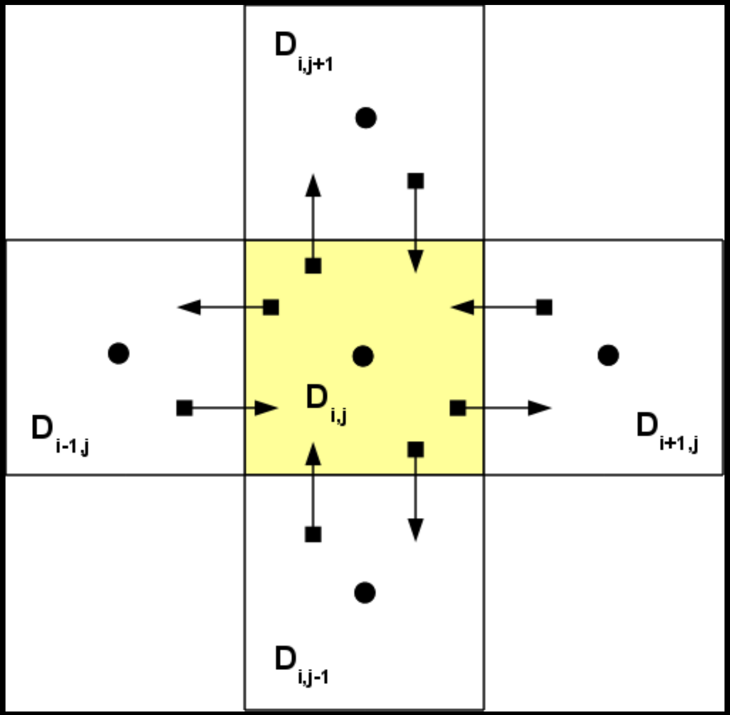
sum over all borders:

$$D_{i,j}^{n+1} = D_{i,j}^{n} + \kappa\, dt\, (D_{i-1,j}^{n} + D_{i+1,j}^{n} + D_{i,j-1}^{n} + D_{i,j+1}^{n} - 4D_{i,j}^{n}) \frac{1}{h^2}$$

unstable!!! :-(

Diffusing the Density

Backtracking the density in the time.



reversion of update rule:

$$D_{i,j}^{n} = D_{i,j}^{n+1} - \kappa\, dt \left(D_{i-1,j}^{n+1} + D_{i+1,j}^{n+1} + D_{i,j-1}^{n+1} + D_{i,j+1}^{n+1} - 4 D_{i,j}^{n+1} \right) \frac{1}{h^2}$$

Solving system of linear equations:

$$\boldsymbol{b} = A x$$

...a linear solver necessary (e.g. Gaussian relaxation)

Moving densities

$$\frac{\partial \rho}{\partial t} = -(\boldsymbol{u}.\nabla)\rho + \kappa \nabla^2 \rho + S$$

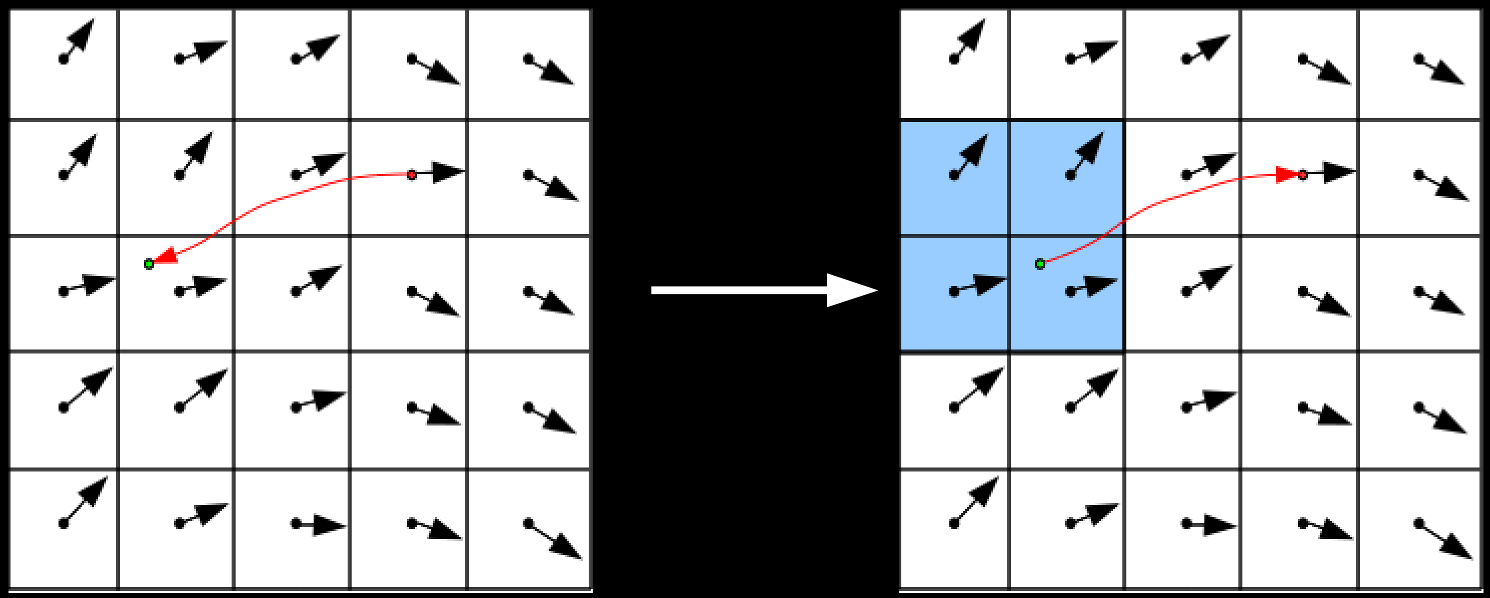
Looking for the stable solution...

stability condition: $\Delta t |\boldsymbol{u}| < h$

Moving Densities

Solving a system of linear equations is impossible...

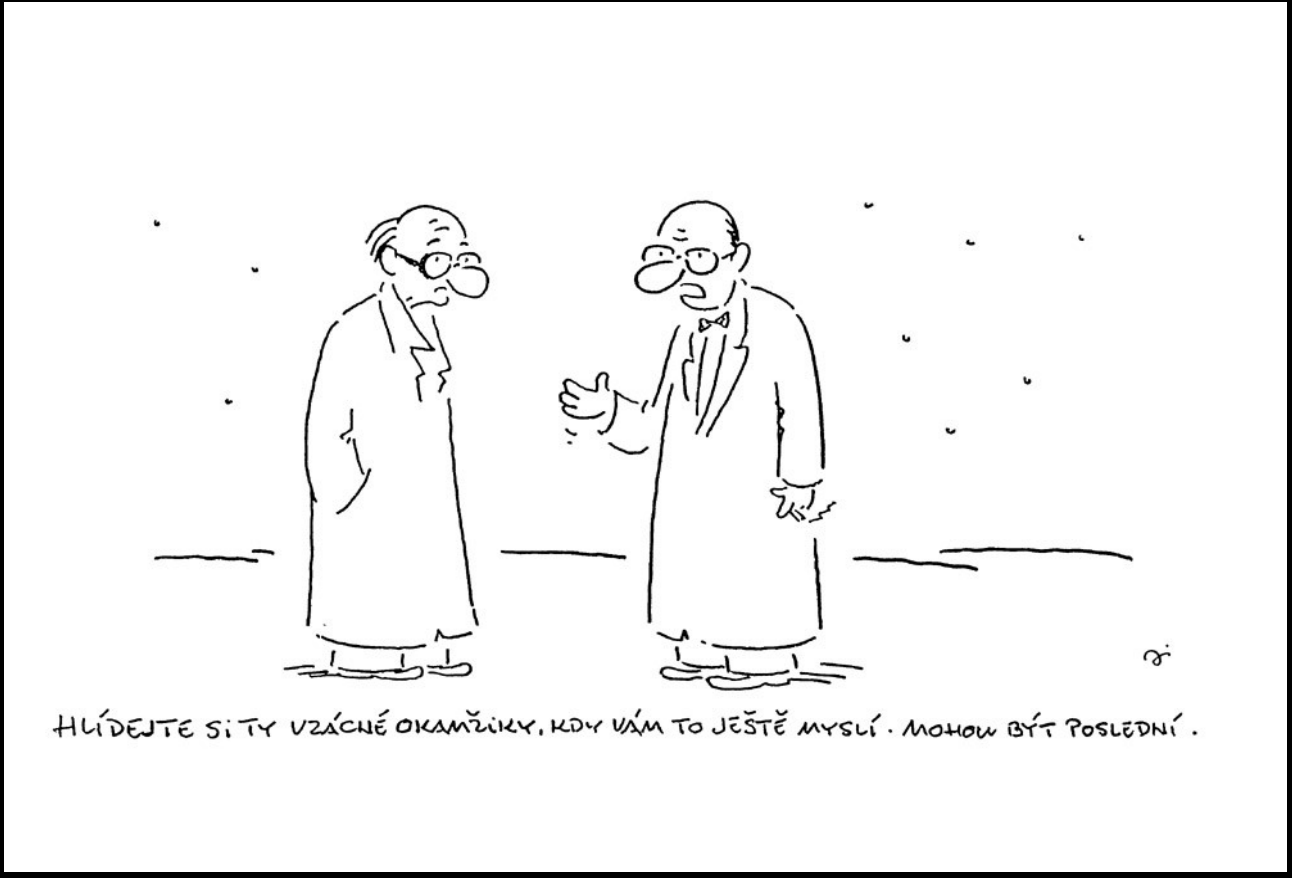
Backtracking particles inside grid



Update of new value interpolating values
of the nearest neighbors

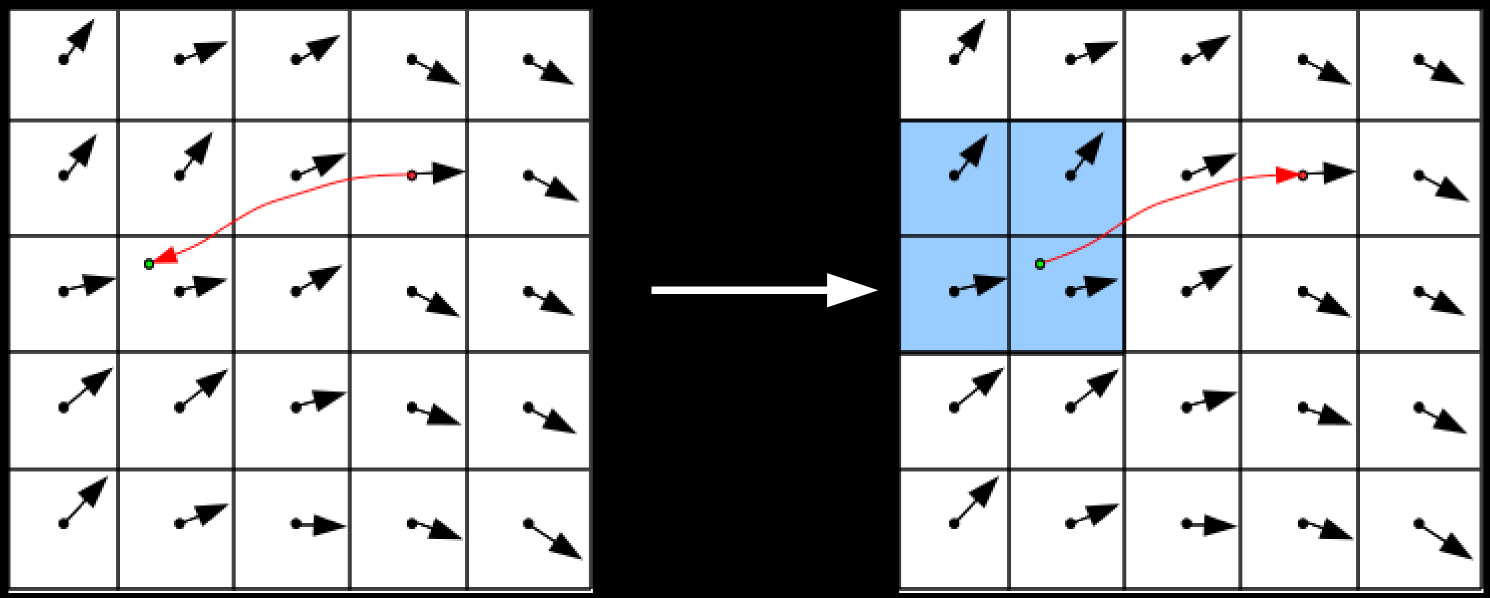
Computing Velocity Field

$$\frac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$



Moving the Velocity Field

Backtracking the particle's velocity as for density before...



Conserving Mass

$$\nabla . \boldsymbol{u} = 0$$

Helmholtz-Hodge Decomposition:

$$\boldsymbol{w} = \boldsymbol{u} + \nabla q$$

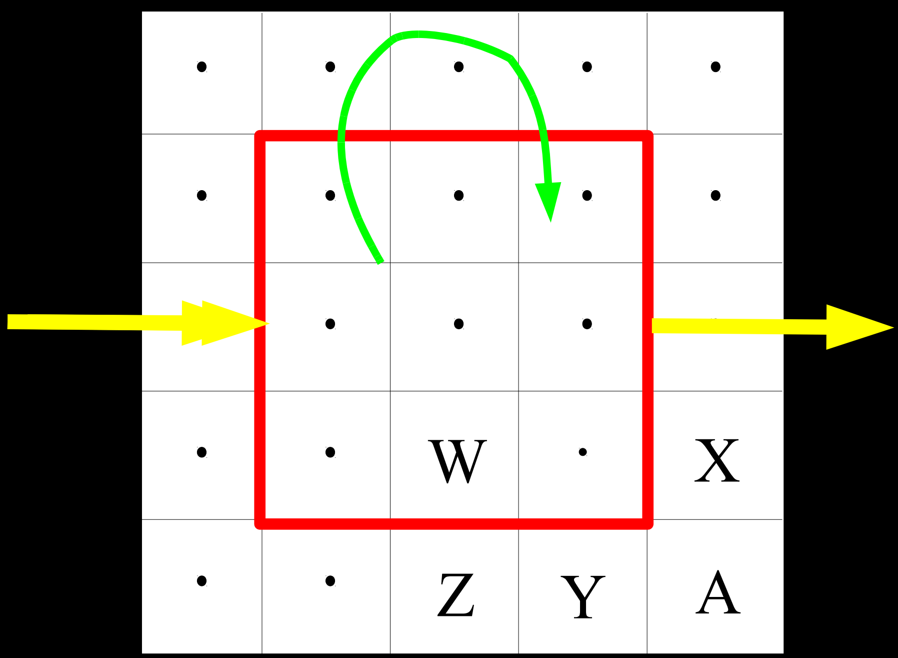
$$\boldsymbol{u} = \boldsymbol{w} - \nabla q$$

$$\nabla . \boldsymbol{w} - \nabla . \nabla q = 0$$

$$\nabla . \boldsymbol{w} - \nabla^2 q = 0$$

Boundary Conditions

simulation inside a closed box
velocity is to be zero on boundaries or
velocity continues on the opposite wall or
… other boundary conditions are possible



A = (X+Y)/2
Z = -W fo rvelocities
Z = W for densities

Example

references

- Jos Stam, "Real-Time Fluid Dynamics for Games". Proceedings of the Game Developer Conference, March 2003.
- Jos Stam, "Stable Fluids", In SIGGRAPH 99 Conference Proceedings, Annual Conference Series, August 1999, 121-128.